UNIVERSITY OF TEXAS AT AUSTIN
Dept. of Electrical and Computer Engineering

Quiz #1

Date: October 4, 1999

Course: EE 313

Name: ______________________________
Last, First

- The exam is scheduled to last 50 minutes.
- Open books and open notes. You may refer to your homework and solution sets.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- Fully justify your answers unless instructed otherwise.

Problem	Point Value	Your Score	Topic
1	30		Differential Equation
2	20		Continuous-Time Convolution
3	30		Tapped Delay Line
4	20		Discrete-Time Stability
Total	100		

Solutions are provided after the exam.

Problem 1.1 Differential Equation. 30 points.
Given the following differential equation

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = f(t)$$

(a) What are the characteristic roots? 5 points.

(b) Find the zero-input response assuming non-zero initial conditions for $y'(0)$ and $y''(0)$. You may leave your answer in terms of C_1 and C_2. 15 points.

(c) Find the zero-input response for the initial conditions $y'(0) = 1$ and $y''(0) = -2$. 10 points.

K-2

Problem 1.2 Continous-Time Convolution. 20 points.

Sketch the following convolutions. On the sketches, clearly label significant points on the t and $y(t)$ axis. You do not have to show intermediate work, but showing intermediate work may qualify for partial credit.

(a) $y(t) = p(t) * p(t)$, where $p(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$. 10 points.

(b) $y(t) = u(-t) * u(-t)$, where $u(t)$ is the unit step function. 10 points.

Problem 1.4 Discrete-Time Stability. 20 points.

Given a linear time-invariant discrete-time system with input $f[k]$ and output $y[k]$ described by the following difference equation

$$y[k] - \frac{3}{2}y[k-1] + Ky[k-2] = f[k]$$

where K is a real-valued parameter,

(a) What are the characteristic roots? 5 points.

(b) For what range of K makes the system stable? 15 points.

Notes

(1) See test questions on attached test

$\boxed{\text{Soln's}}$

(1) (A) $D^2 + 2D + 1$

$\lambda^2 + 2\lambda + 1$

$(\lambda+1)(\lambda+1) \rightarrow \boxed{\lambda = (-1,-1)}$ repeated real root

(B) $\boxed{y_o(t) = c_1 e^{-t} + c_2 t e^{-t}}$

$y_o'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$

$y_o'(0) = -c_1 + c_2$

$y_o''(t) = +c_1 e^{-t} - c_2 e^{-t} - c_2 e^{-t} + c_2 t e^{-t}$

$= c_1 e^{-t} - 2c_2 e^{-t} + c_2 t e^{-t}$

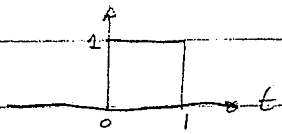
$y_o''(0) = c_1 - 2c_2$

(C) $y_o'(0) = 1 = -c_1 + c_2$; $c_1 = 0$

$y_o''(0) = -2 = c_1 - 2c_2$; $c_2 = 1$

$\boxed{y_o(t) = te^{-t}}$

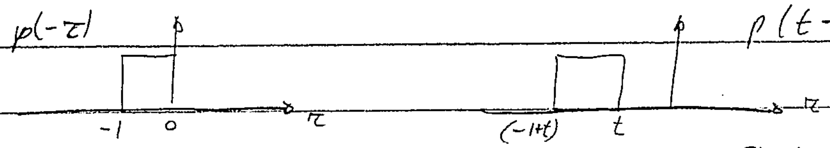
(2) $p(t)$



(A)

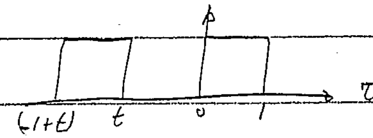
find $y(t) = p(t) * p(t) = \int_{-\infty}^{\infty} p(\tau)\,p(t-\tau)\,d\tau$

$p(-\tau)$ $p(t-\tau)$



FLIP SHIFT

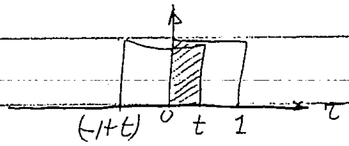
(i)



for $t<0$, no overlap.

$y_i(t) = \int_{\tau=-\infty}^{\tau=0} (0)\,d\tau = 0$

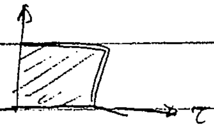
(ii)



for $0 \le t < 1$

$\int_{\tau=0}^{\tau=t} 1 \cdot 1\, d\tau = \tau\Big]_0^t = t$

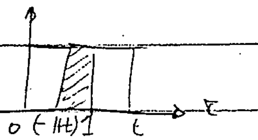
(iii)



for $t=1$ only one instant.

$\int (0) = 0$

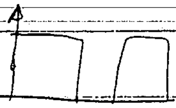
(iv)



for $1 \le t < 2$

$\therefore \int_{\tau=(-1+t)}^{\tau=1} d\tau = \tau\Big]_{(-1+t)}^{1} = 1-(-1+t) = 1+1-t = 2-t$

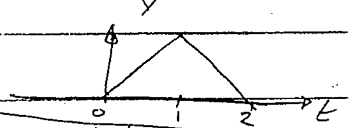
(v)



for $t \ge 2$

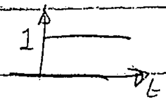
$\int (\cdot) = 0$

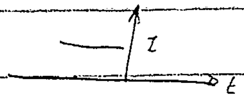
$$y(t) = \begin{cases} 0, & t<0 \\ t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$$



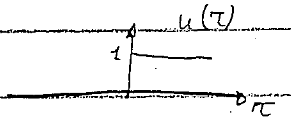
K-6

(2)

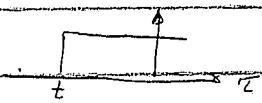
(B) $u(t) =$ 

$u(-t)$ 

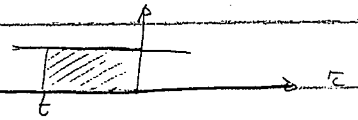
$$y(t) = \underset{f_1(t)}{u(-t)} * \underset{f_1(t)}{u(-t)} = \int_{-\infty}^{\infty} f(\tau)\, f(t-\tau)\, d\tau =$$



FLIP

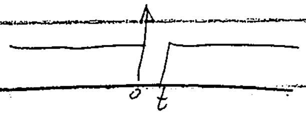


SHIFT

(i) 

for $-\infty < t \leq 0$

$$\int_{\tau=t}^{\tau=0} (1\cdot 1)\, d\tau = \tau\Big]_t^0 = 0 - t = -t$$

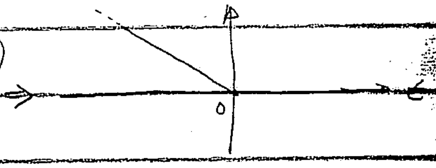
(ii) 

for $t > 0$

no overlap.

$\int (0) = 0$

$$\boxed{y(t) = \begin{cases} -t, & t \leq 0 \\ 0, & t > 0 \end{cases}}$$



3 A finite impulse response

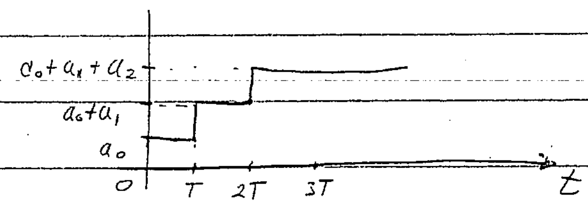
since only have N taps ; $N < \infty$

B $h(t)$ is the output when an impulse $\delta(t)$ is the input. Thus,

$$\boxed{h(t) = \sum_{n=0}^{N-1} a_n \delta(t-nT)}$$

C
$$y(t) = h(t) * \overset{(\text{input})}{u(t)}$$
$$= \sum_{n=0}^{N-1} a_n \delta(t-nT) * u(t)$$
$$\boxed{y(t) = \sum_{n=0}^{N-1} a_n u(t-nT)}$$ by sifting property

D assume $a_0 = a_1 = a_n$ for simplicity of sketching



E From part 'b', see that rise time is $2T$ for $N=3$. Thus,

$$\boxed{T_{cont.} = (N-1)T}$$

4

A $y[k+2] - \frac{3}{2} y[k+1] + K y[k] = f[k+2]$

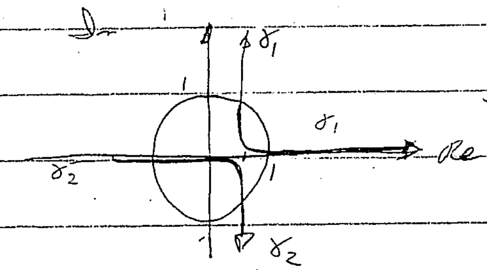
$E^2 - \frac{3}{2} E + K = 0$

$$\boxed{\gamma = \frac{3 \pm \sqrt{9-16K}}{4}, \quad 2 \text{ roots}}$$

B For discrete-time case, it can

(see next page)

general sketch 6



let $S_{\gamma_2} \triangleq$ set of $\gamma_2(K)$ for $|\gamma_2| < 1$

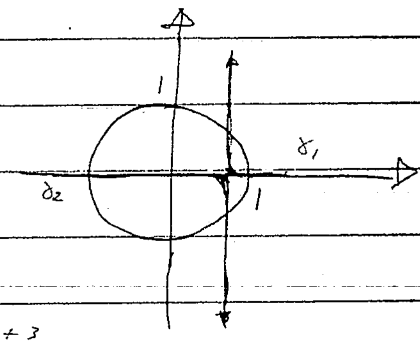
$S_{\gamma_1} \triangleq$ " "

Need $S_{\gamma_1} \cap S_{\gamma_2}$. Need ranges of K such that correspond to this.

① $\gamma = \frac{3}{4} \pm \frac{\sqrt{9-16K}}{4}$

$\gamma_1 = \gamma_2 = \frac{3}{4}$ when $\sqrt{9-16K} = 0$

$K = \frac{9}{16}$



② for $(9-16K) > 0$ $\mathbb{R}$ roots,

γ_1 moving to $+\infty$, γ_2 toward $-\infty$

moving at same rate. away from $+\frac{3}{4}$

so γ_1 will exit unit circle first. So focus on it

$\gamma_1 = \frac{3}{4} + \frac{\sqrt{9-16K}}{4}$, purely $\mathbb{R}$ in this case, and (+)

so no need to take $|\gamma_1|$.

test $\gamma_1 = 1 \rightarrow \frac{1}{4} = \frac{\sqrt{9-16K}}{4}$

$1^2 = (9-16K) \rightarrow K = \frac{-8}{-16} = \frac{1}{2}$

so at $K = \frac{1}{2}$, γ_1 hits circle.

clearly $\gamma_2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$, so marg. stable

③ For $K < \frac{1}{2}$, $|\gamma_1| > 1$ so unstable.

Let $z = \alpha + j\beta$

Then: $(\alpha + j\beta)(\alpha - j\beta) = (\alpha^2 + \beta^2) = z^2 \rightarrow \sqrt{z^2} = |z|$

$[(\alpha^2 + \beta^2)]^{\frac{1}{2}} = |z|$

7

(4) for $(9-16k) < 0$, $\mathbb{C}$ roots.

γ_1 and γ_2 move from $(\frac{3}{4} + j0)$ to $(\frac{3}{4} + jx)$ and $(\frac{3}{4} - jx)$ respectively. They will hit circle at same time since start equidistant vertically from edges. So can look at either.

$$|\gamma_1| = 1 \rightarrow \left|\left(\frac{3}{4} + jx\right)\right| = 1$$

$$\sqrt{\left(\frac{3}{4}\right)^2 + x^2} = 1$$

$$\frac{9}{16} + x^2 = 1$$

$$x^2 = \frac{7}{16}$$

$$x = \pm\sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

So when $\frac{\sqrt{9-16k}}{4} = jx = \frac{\pm\sqrt{7}}{4}$, hit circle.

$$\sqrt{9-16k} = \pm\sqrt{7}\, j$$

$$9 - 16k = -7$$

$$k = \frac{-16}{-16} = 1.$$

@ $k = \frac{1}{8}$

$$\gamma_1 = \left(\frac{3}{4} + \frac{\sqrt{9-16}}{4}\right) = \frac{3}{4} + j\frac{\sqrt{7}}{4}$$

$$\gamma_2 = \left(\frac{3}{4} - \frac{\sqrt{9-16}}{4}\right) = \frac{3}{4} - j\frac{\sqrt{7}}{4}$$

(5) for $k > 1$, outside circle, unstable.

(6) Summary:

$k < \frac{1}{2}$; unstable

$k = \frac{1}{2}$; marg. stable

$\frac{1}{2} < k < 1$, stable

$k = \frac{9}{16} \in (\frac{1}{2}, 1)$, stable, repeated root.

$k > 1$; unstable

$k = 1$; marg. stable, 2 distinct roots on circle